The minimum value and the L^1 norm of the Dirichlet kernel

For each positive integer n, define the function

$$
\begin{aligned}
D_n(\theta) &= 1 + 2\big(\cos\theta + \cos 2\theta + \cdots + \cos n\theta\big) \\
&= e^{-in\theta} + \cdots + e^{-i2\theta} + e^{-i\theta} + e^0 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta}
\end{aligned}
$$

which we call the (nth) Dirichlet kernel. The Dirichlet kernel is a much-studied function in analysis (for example, it arises when considering partial sums of Fourier series). Note that $D_n(\theta)$ is an even function with period 2π.

If $f(\theta)$ is any 2π-periodic function, we define the usual L^p norms

$$\|f(\theta)\|_p = \Big(\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^p \, d\theta\Big)^{1/p}$$

and if $f(\theta)$ is also an even function, we have

$$\|f(\theta)\|_p = \Big(\frac{1}{\pi}\int_0^{\pi} |f(\theta)|^p \, d\theta\Big)^{1/p}.$$

This note is concerned with the minimum value and the L^1 norm of the function $D_n(\theta)$. We prove the following:

$$\min_\theta D_n(\theta) \sim C_0 \cdot n \approx -0.434467n$$

where C_0 is the absolute minimum of $\frac{2\sin t}{t}$, and

$$\|D_n(\theta)\|_1 \sim \frac{4}{\pi^2}\log n \approx 0.405285 \log n$$

where "log" means natural log, and $f(n) \sim g(n)$ means $\frac{f(n)}{g(n)} \to 1$.

These results are not new (indeed, they probably count as "mathematical folklore") but detailed self-contained proofs are sometimes hard to find in the literature, and such proofs can be interesting and instructive analysis exercises.

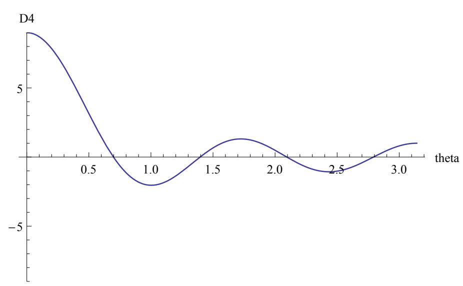
We start by noting that $D = D_n(\theta)$ can be written another way using the following manipulations.

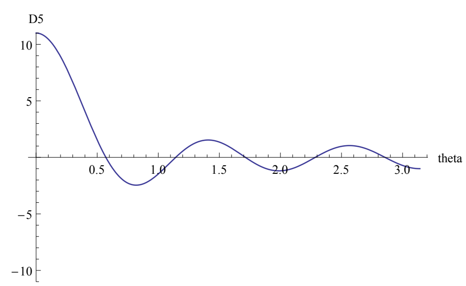
$$\begin{aligned} D &= e^{-in\theta} + \cdots + e^{-i1\theta} + e^{i0\theta} + e^{i1\theta} + \cdots + e^{in\theta} \\ e^{i\theta/2}D &= e^{-i(n-\frac{1}{2})\theta} + \cdots + e^{-i\theta/2} + e^{i\theta/2} + e^{i3\theta/2} + \cdots + e^{i(n+\frac{1}{2})\theta} \\ e^{-i\theta/2}D &= e^{-i(n+\frac{1}{2})\theta} + \cdots + e^{-i3\theta/2} + e^{-i\theta/2} + e^{i\theta/2} + \cdots + e^{i(n-\frac{1}{2})\theta} \\ (e^{i\theta/2} - e^{-i\theta/2})D &= e^{i(n+\frac{1}{2})\theta} - e^{-i(n+\frac{1}{2})\theta} \\ D &= \frac{e^{i(n+\frac{1}{2})\theta} - e^{-i(n+\frac{1}{2})\theta}}{e^{i\theta/2} - e^{-i\theta/2}} = \frac{2i\sin(n+\frac{1}{2})\theta}{2i\sin\frac{\theta}{2}} = \frac{\sin(\frac{2n+1}{2})\theta}{\sin\frac{\theta}{2}}. \end{aligned}$$

If $\theta \in [0, \pi]$, the denominator $\sin\frac{\theta}{2}$ is nonnegative, and the numerator $\sin(\frac{2n+1}{2})\theta$ has sign changes when θ is an integer multiple of $\frac{2\pi}{2n+1}$. We therefore partition the interval $[0, \pi]$ into $n+1$ subintervals

$$\left[0, \frac{2\pi}{2n+1}\right] \cup \left[\frac{2\pi}{2n+1}, \frac{4\pi}{2n+1}\right] \cup \cdots \cup \left[\frac{2(n-1)\pi}{2n+1}, \frac{2n\pi}{2n+1}\right] \cup \left[\frac{2n\pi}{2n+1}, \pi\right]$$

where the first n subintervals each have width $\frac{2\pi}{2n+1}$, and the last subinterval has width $\frac{\pi}{2n+1}$. For illustrative purposes, the graphs of $D_4(\theta)$ and $D_5(\theta)$ are given below. In general, we have $D_n(0) = 2n+1$ and $|D_n(\theta)| \leq 2n+1$.





If we just want to show there exists some negative constant C such that $D_n(\theta)$ always dips below Cn for some θ, we can accomplish this by simply considering $D_n(\frac{3\pi}{2n+1})$.

If $\theta_0 = \frac{3\pi}{2n+1}$, we note the following.

$$\sin\frac{\theta_0}{2} < \frac{\theta_0}{2}$$
$$\frac{1}{\sin\frac{\theta_0}{2}} > \frac{2}{\theta_0}$$
$$D_n(\theta_0) = \frac{\sin(\frac{2n+1}{2})\theta_0}{\sin\frac{\theta_0}{2}} = \frac{-1}{\sin\frac{\theta_0}{2}} < \frac{-2}{\theta_0} = \frac{-2}{3\pi}(2n+1) \sim -\frac{4}{3\pi}n.$$

This might cause someone to conjecture that $\min D_n(\theta)$ is asympotically equal to $-\frac{4}{3\pi}n \approx -0.424413n$, but as mentioned before, the true minimum is closer to $-0.434467n$. The first step toward proving this is to compare $D_n(\theta)$ to a Riemann sum and an integral.

We observe that we have

$$\begin{aligned} D_n(\theta) &= 1 + 2n\sum_{k=1}^{n}\cos\left(n\theta\frac{k}{n}\right)\cdot\frac{1}{n} \\ &= 1 + 2n\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n} \end{aligned}$$

where $f(x) = \cos(n\theta x)$. Then, the sum $\sum_{k=1}^{n} f(\frac{k}{n})\frac{1}{n}$ is a right-endpoint Riemann sum for the integral $\int_0^1 f(x)dx$. We therefore have

$$\begin{aligned} D_n(\theta) &\approx 1 + 2n\int_0^1 f(x)\,dx \\ &= 1 + 2n\int_0^1 \cos(n\theta x)\,dx \\ &= 1 + 2n\cdot\frac{\sin(n\theta)}{n\theta} = 1 + 2n\cdot\frac{\sin t}{t} \end{aligned}$$

which, for a fixed n, is minimized when t minimizes $\frac{\sin t}{t}$. That value of t is approximately 4.49341, suggesting we should choose $\theta \approx 4.49341/n$ (which is a little less than $\frac{3\pi}{2n+1} \approx 4.71239/n$). To make all this more precise, we have to be more careful about comparing the Riemann sum to the integral. (The fact that the function $f(x)$ depends on n makes things nontrivial.)

The integral can be broken into smaller integrals

$$\int_0^1 f(x)\, dx = \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} f(x)\, dx$$

and each of these smaller integrals can be written as

$$\int_{(k-1)/n}^{k/n} f(x)\, dx = f(x_k^*) \cdot \frac{1}{n} \qquad \text{where } x_k^* \in [\tfrac{k-1}{n}, \tfrac{k}{n}].$$

We therefore have $\left|\frac{k}{n} - x_k^*\right| \leq \frac{1}{n}$. We also have

$$\frac{f(\frac{k}{n}) - f(x_k^*)}{\frac{k}{n} - x_k^*} = f'(\xi) \qquad \text{for some } \xi \text{ between } x_k^* \text{ and } \tfrac{k}{n}.$$

This implies

$$\begin{aligned}
\left|\frac{f(\frac{k}{n}) - f(x_k^*)}{\frac{k}{n} - x_k^*}\right| &= |f'(\xi)| = |-n\theta \sin(n\theta\xi)| \leq n\theta \\
\left|f\left(\frac{k}{n}\right) - f(x_k^*)\right| &\leq n\theta \left|\frac{k}{n} - x_k^*\right| \leq n\theta \cdot \frac{1}{n} = \theta \\
\left|f\left(\frac{k}{n}\right) \cdot \frac{1}{n} - f(x_k^*) \cdot \frac{1}{n}\right| &\leq \frac{\theta}{n}.
\end{aligned}$$

That is, $f(\frac{k}{n})\frac{1}{n}$ is within $\frac{\theta}{n}$ of $f(x_k^*)\frac{1}{n} = \int_{(k-1)/n}^{k/n} f(x)dx$. If we then sum from $k = 1$ to n, we get

$$\begin{aligned}
\int_0^1 f(x)\, dx - \theta \leq \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} &\leq \int_0^1 f(x)\, dx + \theta \\
\frac{\sin(n\theta)}{n\theta} - \theta \leq \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot \frac{1}{n} &\leq \frac{\sin(n\theta)}{n\theta} + \theta \\
\frac{1}{n} + \frac{2\sin(n\theta)}{n\theta} - 2\theta \leq \frac{D_n(\theta)}{n} &\leq \frac{1}{n} + \frac{2\sin(n\theta)}{n\theta} + 2\theta.
\end{aligned}$$

To prove our claim about $\min_\theta D_n(\theta)$, we have to prove that

$$\frac{\min_\theta D_n(\theta)}{n} \to C_0 \approx -0.434467.$$

We will prove this by proving the following.

Proposition: Given $\varepsilon > 0$, there exists N such that if $n > N$, then
(Claim 1) $D_n(\theta)/n \geq C_0 - \varepsilon$ for all θ,
(Claim 2) $D_n(\theta)/n \leq C_0 + \varepsilon$ for some θ.

To prove Claim 1, we use different arguments for different values of θ. Claim 1 is trivially true if θ belongs to either of the intervals $[0, \frac{2\pi}{2n+1}]$ or $[\frac{4\pi}{2n+1}, \frac{6\pi}{2n+1}]$, because $\sin(\frac{2n+1}{2})\theta$ is nonnegative there and hence so is $D_n(\theta)$. Next, we consider $\theta \in [\frac{6\pi}{2n+1}, \pi]$. Using the fact that $\sin t \geq \frac{2}{\pi} t$ on the interval $[0, \frac{\pi}{2}]$, we have

$$\sin\frac{\theta}{2} \geq \frac{\theta}{\pi}$$

$$\frac{1}{\sin\frac{\theta}{2}} \leq \frac{\pi}{\theta} < \frac{2n+1}{6}$$

$$|D_n(\theta)| = \frac{\left|\sin(\frac{2n+1}{2})\theta\right|}{\sin\frac{\theta}{2}} \leq \frac{1}{\sin\frac{\theta}{2}} < \frac{2n+1}{6}$$

$$\left|\frac{D_n(\theta)}{n}\right| < \frac{1}{3} + \frac{1}{6n}.$$

Therefore if $n \geq 2$ we have

$$\left|\frac{D_n(\theta)}{n}\right| < \frac{1}{3} + \frac{1}{12} = \frac{5}{12} \approx 0.416667$$

$$C_0 - \varepsilon < C_0 < -\frac{5}{12} < \frac{D_n(\theta)}{n} < \frac{5}{12}.$$

We now must show that Claim 1 is true for all θ in the interval $[\frac{2\pi}{2n+1}, \frac{4\pi}{2n+1}]$, and we must show that $D_n(\theta)/n \leq C_0 + \varepsilon$ for some θ. This will finish the proof of the above proposition.

From before, we have

$$\frac{1}{n} + \frac{2\sin(n\theta)}{n\theta} - 2\theta \leq \frac{D_n(\theta)}{n} \leq \frac{1}{n} + \frac{2\sin(n\theta)}{n\theta} + 2\theta.$$

If $\theta \leq \frac{4\pi}{2n+1}$, then this implies

$$\frac{D_n(\theta)}{n} \geq \frac{1}{n} + \frac{2\sin(n\theta)}{n\theta} - 2\theta \geq \frac{1}{n} + C_0 - \frac{8\pi}{2n+1}$$

which will be above $C_0 - \varepsilon$ if n is large enough.

Next, let t_0 be the t that minimizes $\frac{\sin t}{t}$ (so $t_0 \approx 4.49341$), and choose $\theta = t_0/n$. We then have

$$\frac{D_n(\theta)}{n} \le \frac{1}{n} + \frac{2\sin t_0}{t_0} + \frac{2t_0}{n} = \frac{1}{n} + C_0 + \frac{2t_0}{n}$$

which will be below $C_0 + \varepsilon$ if n is large enough.

This completes our proof that $\min_\theta D_n(\theta) \sim C_0 \cdot n \approx -0.434467n$. Next, we proceed with our analysis of

$$\|D_n(\theta)\|_1 = \frac{1}{\pi}\int_0^\pi |D_n(\theta)|\, d\theta.$$

Following our earlier remarks about sign changes of $D_n(\theta)$ and $\sin(\frac{2n+1}{2})\theta$, we write

$$\pi \|D_n(\theta)\|_1 = \sum_{k=1}^{n} \int_{2(k-1)\pi/(2n+1)}^{2k\pi/(2n+1)} |D_n(\theta)|\, d\theta + \int_{2n\pi/(2n+1)}^{\pi} |D_n(\theta)|\, d\theta.$$

We now assemble some various lemmas that will be useful.

Lemma 1. If $a > 0$ is real and k is an integer, we have

$$\int_{(k-1)\pi/a}^{k\pi/a} |\sin(a\theta)|\, d\theta = \frac{2}{a}.$$

This lemma is straightforward, and the proof is omitted. We will use this lemma with $a = \frac{2n+1}{2}$, in which case it says

$$\int_{2(k-1)\pi/(2n+1)}^{2k\pi/(2n+1)} \left|\sin\left(\frac{2n+1}{2}\right)\theta\right| d\theta = \frac{4}{2n+1}.$$

Lemma 2. If $0 < t < \frac{\pi}{2}$, we have

$$\frac{1}{t} \le \frac{1}{\sin t} \le \frac{1}{t} + C_1 \cdot t$$

where $C_1 = \frac{4}{24-\pi^2} \approx 0.283078$.

Proof of Lemma 2: For positive t, we have

$$t - \frac{t^3}{3!} \leq \sin t \leq t$$

which implies

$$\begin{aligned} \frac{1}{t} \leq \frac{1}{\sin t} \leq \frac{1}{t - \frac{t^3}{6}} &= \frac{6}{6t - t^3} \\ &= \frac{6 - t^2}{6t - t^3} + \frac{t^2}{6t - t^3} \\ &= \frac{1}{t} + \frac{t}{6 - t^2} \end{aligned}$$

which means that if $t \leq \frac{\pi}{2}$ then we further have

$$\frac{1}{\sin t} \leq \frac{1}{t} + \frac{t}{6 - t^2} \leq \frac{1}{t} + \frac{t}{6 - (\frac{\pi}{2})^2} = \frac{1}{t} + \frac{4t}{24 - \pi^2}.$$

Lemma 3. If we define

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

then $H_n - \log n$ can be bounded between two constants (in fact, it approaches a constant). For example, we have

$$1 - \log 2 \leq H_n - \log n \leq 1 \qquad \text{for all } n.$$

The proof of Lemma 3 is reasonably straightforward and is hence omitted. Essentially, we regard H_n as a Riemann sum and compare to an integral.

With these lemmas stated, we are in a position to estimate $\|D_n(\theta)\|_1$. Recall that we expressed $\pi \|D_n(\theta)\|_1$ as a sum of $n + 1$ integrals

$$\pi \|D_n(\theta)\|_1 = \sum_{k=1}^{n} \int_{2(k-1)\pi/(2n+1)}^{2k\pi/(2n+1)} |D_n(\theta)| \, d\theta + \int_{2n\pi/(2n+1)}^{\pi} |D_n(\theta)| \, d\theta.$$

For two of those $n + 1$ integrals, we just use trivial bounds. We know

$$0 \leq |D_n(\theta)| \leq 2n + 1.$$

This implies that we have

$$0 \le \int_0^{2\pi/(2n+1)} |D_n(\theta)|\, d\theta \le (2n+1) \cdot \frac{2\pi}{2n+1} = 2\pi,$$
$$0 \le \int_{2n\pi/(2n+1)}^{\pi} |D_n(\theta)|\, d\theta \le (2n+1) \cdot \frac{\pi}{2n+1} = \pi.$$

Next, we want upper and lower bounds for the $n-1$ integrals of the form

$$\int_{2(k-1)\pi/(2n+1)}^{2k\pi/(2n+1)} |D_n(\theta)|\, d\theta$$

where $2 \le k \le n$. Note that we have

$$|D_n(\theta)| = \left| \frac{\sin(\frac{2n+1}{2})\theta}{\sin \frac{\theta}{2}} \right| = \frac{\left|\sin(\frac{2n+1}{2})\theta\right|}{\sin \frac{\theta}{2}}.$$

Then, for θ in the above interval, we have

$$\frac{(k-1)\pi}{2n+1} \le \frac{\theta}{2} \le \frac{k\pi}{2n+1}$$
$$\sin \frac{(k-1)\pi}{2n+1} \le \sin \frac{\theta}{2} \le \sin \frac{k\pi}{2n+1}$$
$$\frac{1}{\sin \frac{k\pi}{2n+1}} \le \frac{1}{\sin \frac{\theta}{2}} \le \frac{1}{\sin \frac{(k-1)\pi}{2n+1}}$$

and then applying Lemma 2 gives us

$$\frac{2n+1}{k\pi} \le \frac{1}{\sin \frac{\theta}{2}} \le \frac{2n+1}{(k-1)\pi} + C_1 \cdot \frac{(k-1)\pi}{2n+1}.$$

Multiplying by $\left|\sin(\frac{2n+1}{2})\theta\right|$ then gives us

$$\frac{2n+1}{k\pi} \left|\sin\left(\frac{2n+1}{2}\right)\theta\right| \le |D_n(\theta)| \le \left(\frac{2n+1}{(k-1)\pi} + C_1 \cdot \frac{(k-1)\pi}{2n+1}\right) \left|\sin\left(\frac{2n+1}{2}\right)\theta\right|.$$

We then integrate from $\theta = \frac{2(k-1)\pi}{2n+1}$ to $\frac{2k\pi}{2n+1}$ and use Lemma 1 with $a = \frac{2n+1}{2}$. This gives us

$$\frac{2n+1}{k\pi} \cdot \frac{4}{2n+1} \le \int_{2(k-1)\pi/(2n+1)}^{2k\pi/(2n+1)} |D_n(\theta)|\, d\theta \le \left(\frac{2n+1}{(k-1)\pi} + C_1 \cdot \frac{(k-1)\pi}{2n+1}\right) \cdot \frac{4}{2n+1}$$

which simplifies to

$$\frac{4}{k\pi} \le \int_{2(k-1)\pi/(2n+1)}^{2k\pi/(2n+1)} |D_n(\theta)|\, d\theta \le \frac{4}{(k-1)\pi} + C_1 \cdot \frac{4(k-1)\pi}{(2n+1)^2}.$$

We will get upper and lower bounds for $\pi \|D_n(\theta)\|_1$ if we sum the above from $k = 2$ to n (and use the previously mentioned trivial bounds on the other integrals). This gives us

$$\pi \|D_n(\theta)\|_1 \ge \sum_{k=2}^{n} \frac{4}{k\pi},$$
$$\pi \|D_n(\theta)\|_1 \le 3\pi + \sum_{k=2}^{n} \left(\frac{4}{(k-1)\pi} + C_1 \cdot \frac{4(k-1)\pi}{(2n+1)^2} \right).$$

Next, with the help of Lemma 3, we have

$$\sum_{k=2}^{n} \frac{4}{k\pi} = \frac{4}{\pi}(H_n - 1) \ge \frac{4}{\pi}(\log n - \log 2),$$
$$\sum_{k=2}^{n} \frac{4}{(k-1)\pi} = \frac{4}{\pi}(H_{n-1}) \le \frac{4}{\pi}(\log(n-1) + 1) < \frac{4}{\pi}(\log n + 1).$$

Note that we also have

$$\begin{aligned} \sum_{k=2}^{n} C_1 \frac{4(k-1)\pi}{(2n+1)^2} &= \frac{4C_1\pi}{(2n+1)^2} \sum_{k=2}^{n} (k-1) \\ &= \frac{C_1\pi}{(n+\frac{1}{2})^2} \cdot \frac{n(n-1)}{2} < \frac{C_1\pi}{2}. \end{aligned}$$

This means that we have upper and lower bounds for $\pi \|D_n(\theta)\|_1$ that are both of the form $\frac{4}{\pi} \log n \pm C$. This completes our proof that

$$\|D_n(\theta)\|_1 \sim \frac{4}{\pi^2} \log n$$

and in fact, our argument proves the slightly stronger result that

$$\|D_n(\theta)\|_1 = \frac{4}{\pi^2} \log n + O(1).$$