A note on epsilon-delta proofs

"Epsilon-delta" proofs can be confusing to calculus students when they first encounter them. To construct such a proof, one of course has to show that for every positive ε, there exists a δ that "works" for that epsilon. Perhaps part of what bothers some beginning students is that the *trial-and-error* process of *finding* a sufficient δ is distinct from the *logical* process of *proving* that the choice of δ works. Often, we do "scratch work" that we don't include in the proof, and so the finished proof can be completely mathematically valid but contain a seemingly "unmotivated" statement along the lines of "let $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{7}\}$."

To some extent, though, one can *combine the two tasks* of finding a δ and proving that the δ works, and I have a preference for epsilon-delta proofs that are written with this philosophy in mind. As a professional mathematician, I prefer a proof (as long as it is valid) to contain relatively few statements that appear unmotivated. Furthermore, it is my hope that this approach to epsilon-delta proofs may be helpful to students who are new to the topic.

I will illustrate what I mean by including two complete self-contained rigorous proofs of the following statement. The first is structured in a way that in my experience, I would consider "typical" for an epsilon-delta proof in a course or textbook introducing the topic, whereas the second proof is structured in the way that I prefer.

Problem. Give an epsilon-delta proof of the statement

$$\lim_{x\to 3} \frac{x^2-5}{x-2} = 4.$$

Proof 1. Let $\varepsilon > 0$. We must find δ such that

$$0 < |x-3| < \delta \Longrightarrow \left|\frac{x^2-5}{x-2} - 4\right| < \varepsilon.$$

Choose $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{5}\}$. If $0 < |x-3| < \delta$, we then have $|x-3| < \frac{\varepsilon}{5}$, and also $-\frac{1}{2} < x-3 < \frac{1}{2}$ which implies $\frac{1}{2} < x-2 < \frac{3}{2}$ and $\frac{3}{2} < x-1 < \frac{5}{2}$, which in turn implies $\left|\frac{1}{x-2}\right| < 2$ and $|x-1| < \frac{5}{2}$. We then have

$$\begin{aligned}
|x-3| \cdot \left|\frac{1}{x-2}\right| \cdot |x-1| &< \frac{\varepsilon}{5} \cdot 2 \cdot \frac{5}{2} \\
\Longrightarrow \left|\frac{(x-3)(x-1)}{x-2}\right| &< \varepsilon \\
\Longrightarrow \left|\frac{x^2-4x+3}{x-2}\right| &< \varepsilon \\
\Longrightarrow \left|\frac{x^2-5-4(x-2)}{x-2}\right| &< \varepsilon \\
\Longrightarrow \left|\frac{x^2-5}{x-2}-4\right| &< \varepsilon
\end{aligned}$$

as required.

Proof 2. Let $\varepsilon > 0$. We must find δ such that

$$0 < |x-3| < \delta \Longrightarrow \left|\frac{x^2-5}{x-2}-4\right| < \varepsilon.$$

We first observe the chain of equalities

$$\begin{aligned}
\left|\frac{x^2-5}{x-2}-4\right| &= \left|\frac{x^2-5-4(x-2)}{x-2}\right| && (1) \\
&= \left|\frac{x^2-4x+3}{x-2}\right| && (2) \\
&= \left|\frac{(x-3)(x-1)}{x-2}\right| && (3) \\
&= \left|\frac{x-1}{x-2}\right| \cdot |x-3|\,. && (4)
\end{aligned}$$

We then observe that if $|x-3| < \frac{1}{2}$, i.e., if $-\frac{1}{2} < x-3 < \frac{1}{2}$, then we have

$$\begin{aligned}
&\frac{1}{2} < x-2 < \frac{3}{2} \quad \text{and} \quad \frac{3}{2} < x-1 < \frac{5}{2} && (5) \\
\Longrightarrow &\left|\frac{1}{x-2}\right| < 2 \quad \text{and} \quad |x-1| < \frac{5}{2} && (6) \\
\Longrightarrow &\left|\frac{x-1}{x-2}\right| < 5. && (7)
\end{aligned}$$

We now choose $\delta = \min\{\frac{1}{2}, \frac{\varepsilon}{5}\}$. Then if $0 < |x-3| < \delta$, we have $|x-3| < \frac{1}{2}$, implying $\left|\frac{x-1}{x-2}\right| < 5$ by (7), as well as $|x-3| < \frac{\varepsilon}{5}$. Taking the product gives

$$\left|\frac{x-1}{x-2}\right| \cdot |x-3| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$$

which by (1)–(4) is equivalent to

$$\left|\frac{x^2-5}{x-2} - 4\right| < \varepsilon$$

as required.

Of course, taste is subjective, and some may consider the first proof to be more elegant, since it begins with the (perhaps unmotivated) choice of δ, and basically consists of nothing but a unidirectional chain of implications, beginning with the choice of delta, and ending with the required inequality involving epsilon. Nevertheless, I find the second proof to be more "honest." In any case, note that the two proofs are essentially the same length, and both are completely rigorous!