Merit Factor of Chu Sequences and Best Merit Factor of Polyphase Sequences

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1 Abstract

Chu sequences are a family of polyphase sequences that have perfect periodic autocorrelations and good aperiodic autocorrelations. It has previously been proved that the maximum offpeak (aperiodic) autocorrelation (in absolute value) of the Chu sequence of length n is asymptotically equal to $0.480261\sqrt{n}$. It has also been empirically observed that the merit factor of Chu sequences appears to grow like a constant times $\sqrt{n}$. In this note, we provide an analytic proof that the merit factor of the Chu sequence of length n is bounded below by a constant multiple of $\sqrt{n}$ for all n. To the author's knowledge, this is the first time a family of polyphase sequences of all lengths has been proved to have merit factor growing at least like order $\sqrt{n}$.

2 Preliminaries

A **complex sequence** of length n is a finite sequence

$$S = (s_0, s_1, \dots, s_{n-1}),$$

where for each j, we have $s_j \in \mathbb{C}$ and $|s_j| = 1$. The sequence is called a **polyphase sequence** if there exists a positive integer M such that each s_j is an Mth root of 1.

For $k = 0, 1, \dots, n-1$, we define the **aperiodic autocorrelations** of S,

$$c_k = \sum_{j=0}^{n-k-1} \overline{s_j} s_{j+k},$$

and the **periodic autocorrelations** of S,

$$\gamma_k = \sum_{j=0}^{n-1} \overline{s_j} s_{j+k},$$

where the bar denotes complex conjugation. In the definition of γ_k, the addition in the subscript is modulo n.

We have $c_0 = \gamma_0 = n$, which we call the "trivial" autocorrelations. It is not hard to show that $\gamma_k = c_k + \overline{c_{n-k}}$. Also note that $|c_{n-1}| = |\overline{s_0} s_{n-1}| = 1$.

We can either seek complex sequences with periodic autocorrelations near zero, or with aperiodic autocorrelations near zero. If $\gamma_k = 0$ for all $k \neq 0$, we say S is a perfect sequence. If $|c_k| \leq 1$ for all $k \neq 0$, we say S is a generalized Barker sequence. If we further have $s_j^M = 1$ for all M, we say S is a polyphase Barker sequence.

Given a complex sequence S, two natural measures of the closeness to zero of the aperiodic autocorrelations are as follows:

$$P(S) = \max_{0<k<n-1} |c_k|,$$
$$T(S) = \sum_{0<k\leq n-1} |c_k|^2,$$

which we respectively call the **peak sidelobe level** (PSL) and **total sidelobe energy** (TSE). (Some authors define the TSE to be twice our value, since they also define $c_{-k} = \overline{c_k}$.) One can also define the **merit factor** of the sequence S, which in our notation is

$$F(S) = \frac{n^2}{2 \cdot T(S)}.$$

We define three functions of n:

$$P_{\min}(n) = \min_S P(S),$$
$$T_{\min}(n) = \min_S T(S),$$
$$F_{\max}(n) = \max_S F(S) = \frac{n^2}{2 \cdot T_{\min}(S)},$$

where the extremum is taken over all complex sequences of length n. These are mathematically well-defined (the "space" of all complex sequences of length n is the product of n copies of the unit circle, and is hence compact) but explicit computation of the extrema appears very difficult.

It would be of interest to have good bounds for the growth rates of the functions $P_{\min}(n)$ and $T_{\min}(n)$.

If S is a generalized Barker sequence of length n, then $P(S) \le 1$ and $T(S) \le n-1$. Therefore, if there is an infinite family of generalized Barker sequences, their merit factor would be at least $n^2/(2(n-1)) \sim n/2$.

Polyphase Barker sequences have been found for all lengths up to N, where the value of N has been gradually increasing. For example, Friese [4] found polyphase Barker sequences of all lengths up to 36, but conjectured that they do not exist for significantly higher lengths. However, Borwein and Ferguson [2] found polyphase Barker sequences of all lengths up to 63, and Nunn and Coxson [7] found polyphase Barker sequences of lengths 64 to 70, as well as 72, 76, and 77. It is unknown whether there exist generalized Barker sequences for every length n.

Thus, we have $P_{\min}(n) \le 1$ for all $n \le 70$, and possibly for larger n. However, the best known asymptotic upper bound in the literature appears to be of the form $P_{\min}(n) = O(\sqrt{n})$. The existence of an infinite family of polyphase sequences with PSL bounded above by $O(\sqrt{n})$ was shown by Turyn [9], and the fact that the Chu sequence of length n has PSL asymptotic to $0.480261\sqrt{n}$ was shown by Mow and Li [5].

It has been empirically observed [1, 8] that Chu sequences, and some related sequences, appear to have merit factor that grows like a constant times $\sqrt{n}$, or equivalently, their TSE appears to grow like a constant times $n^{3/2}$.

The main purpose of this note is to provide an analytic proof that, if S is the Chu sequence of length n, we have a bound of the form

$$T(S) = \sum_{k \ne 0} |c_k|^2 \le Bn^{3/2}$$

that holds for all n (where B is constant). This implies $T_{\min}(n) \le Bn^{3/2}$.

3 Proof of Main Result

Main result. Let S be the Chu sequence of length n, as defined in [3]. Then the total sidelobe energy of S is bounded by

$$\begin{aligned} T(S) &\le \frac{8}{3\pi^{3/2}} \cdot n^{3/2} + O(n) \\ &\approx 0.4788989922 n^{3/2} + O(n) \end{aligned}$$

and the merit factor of S is bounded below by

$$F(S) = \frac{n^2}{2T(S)} \ge C\sqrt{n}$$

where C is asymptotic to $3\pi^{3/2}/16 \approx 1.0440615$ when $n \to \infty$.

Proof. As in [3], for each positive integer n, the **Chu sequence** of length n is the sequence $S = (s_0, s_1, \dots, s_{n-1})$, where

$$s_j = \begin{cases} \mathbf{e}\Big(\dfrac{j^2}{2n}\Big) & \text{if } n \text{ is even,} \\ \mathbf{e}\Big(\dfrac{j(j+1)}{2n}\Big) & \text{if } n \text{ is odd,} \end{cases}$$

where $\mathbf{e}(t)$ is shorthand for $e^{i2\pi t}$.

Assume $k \ne 0$. As proved in [3], the Chu sequences satisfy $\gamma_k = 0$ for all k. This implies that $|c_{n-k}| = |c_k|$. Now note that in general, c_{n-k} is a sum of k terms of absolute value at most 1. Therefore $|c_k| = |c_{n-k}| \le k$.

Using some relatively straightforward manipulations (see, e.g., the beginning of Section IV in [6], or the proof of Theorem 2 in [10]) one can show that the Chu sequences satisfy

$$|c_k| = \left| \frac{\sin\left(\dfrac{\pi k^2}{n}\right)}{\sin\left(\dfrac{\pi k}{n}\right)} \right| .$$

For convenience, we define the **half-energy**

$$H(S) = \sum_{0<k<\lceil n/2 \rceil} |c_k|^2 .$$

Using the relation $|c_{n-k}| = |c_k|$, one can see that

$$T(S) = \begin{cases} 2 \cdot H(S) & \text{if } n \text{ is odd,} \\ 2 \cdot H(S) + \left|c_{n/2}\right|^2 & \text{if } n \text{ is even.} \end{cases}$$

Now note that

$$\left|c_{n/2}\right| = \left| \frac{\sin\left(\frac{\pi n}{4}\right)}{\sin\left(\frac{\pi}{2}\right)} \right| = \left|\sin\left(\frac{\pi n}{4}\right)\right| \leq 1.$$

The key is now to break the sum $H(S)$ into two pieces

$$H(S) = \sum_{1 \leq k \leq \lceil An^{1/2} \rceil} |c_k|^2 + \sum_{\lceil An^{1/2} \rceil < k < \lceil n/2 \rceil} |c_k|^2$$

where A is a fixed positive real number yet to be determined.

Let $m = \lceil An^{1/2} \rceil$ and let $m_2 = \lceil n/2 \rceil - 1$. Since $|c_k| \leq k$, we have

$$\sum_{1 \leq k \leq m} |c_k|^2 \leq \sum_{1 \leq k \leq m} k^2 = \frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6}.$$

We now consider

$$\sum_{m < k < \lceil n/2 \rceil} |c_k|^2 = \sum_{m < k \leq m_2} \sin^2\left(\frac{\pi k^2}{n}\right) \csc^2\left(\frac{\pi k}{n}\right) \leq \sum_{m < k \leq m_2} \csc^2\left(\frac{\pi k}{n}\right).$$

This latter sum is equal to

$$\sum_{m < k \leq m_2} n \csc^2\left(\pi \cdot \frac{k}{n}\right) \cdot \frac{1}{n}$$

which can be regarded as a Riemann sum. If the interval of real numbers $[\frac{m}{n}, \frac{m_2}{n}]$ is divided into subintervals of width $1/n$, then as k ranges through the integer values $m < k \leq m_2$, the numbers k/n are the right endpoints of the subintervals.

Note that $m_2 \leq (n-1)/2 < n/2$, and that $m > 0$. Therefore $[\frac{m}{n}, \frac{m_2}{n}]$ is a subset of the interval $(0, \frac{1}{2})$, on which the function $\csc^2(\pi t)$ is strictly decreasing. It follows that we have

$$\sum_{m<k\leq m_2} n \csc^2\left(\pi \cdot \frac{k}{n}\right) \cdot \frac{1}{n} \;\leq\; \int_{m/n}^{m_2/n} n \csc^2(\pi t)\, dt$$

and the integral evaluates to

$$\left[\frac{-n\cot(\pi t)}{\pi}\right]_{m/n}^{m_2/n} \;=\; \frac{n}{\pi}\left[\cot\left(\frac{\pi m}{n}\right) - \cot\left(\frac{\pi m_2}{n}\right)\right]. \tag{1}$$

On the interval $(0, \frac{\pi}{2})$, the function $\cot x$ is positive and satisfies $\cot x < 1/x$. The quantity (1) can therefore be bounded above by

$$\frac{n}{\pi}\cot\left(\frac{\pi m}{n}\right) < \frac{n}{\pi} \cdot \frac{n}{\pi m} = \frac{n^2}{\pi^2 m}.$$

Combining our bounds, we conclude

$$H(S) < \frac{m^3}{3} + \frac{m^2}{2} + \frac{m}{6} + \frac{n^2}{\pi^2 m}.$$

Since $A\sqrt{n} \leq m < A\sqrt{n} + 1$, we conclude

$$H(S) < \frac{(A\sqrt{n}+1)^3}{3} + \frac{(A\sqrt{n}+1)^2}{2} + \frac{A\sqrt{n}+1}{6} + \frac{n^2}{\pi^2 A\sqrt{n}} \tag{2}$$

which is a polynomial in $\sqrt{n}$ of degree 3. The coefficient of the dominant power $n^{3/2}$ is

$$\frac{A^3}{3} + \frac{1}{\pi^2 A}.$$

Using elementary calculus, we find that the positive A that minimizes this quantity is

$$A = \frac{1}{\sqrt{\pi}}$$

and the minimum value itself is

$$\frac{A^3}{3} + \frac{1}{\pi^2 A} = \frac{4}{3\pi^{3/2}} \approx 0.2394494961.$$

Choosing $A = 1/\sqrt{\pi}$, we then have an upper bound of the form

$$H(S) < \frac{4}{3\pi^{3/2}} \cdot n^{3/2} + O(n)$$

which implies

$$\begin{aligned} T(S) &\leq 2 \cdot H(S) + 1 \\ &< \frac{8}{3\pi^{3/2}} \cdot n^{3/2} + O(n) \\ &\approx 0.4788989922 \cdot n^{3/2} + O(n). \end{aligned}$$

This implies that the merit factor $F(S)$ asymptotically grows at least like

$$\frac{n^2}{2 \cdot \dfrac{8}{3\pi^{3/2}} \cdot n^{3/2}} \;=\; \frac{3\pi^{3/2}}{16}\sqrt{n} \;\approx\; 1.044061500\sqrt{n}.$$

If we seek explicit bounds that are valid for all n, we can start by evaluating the bound (2) when $A = 1/\sqrt{\pi}$. This gives

$$H(S) < \frac{4}{3\pi^{3/2}} \cdot n^{3/2} + \frac{3}{2\pi} \cdot n + \frac{13}{6\sqrt{\pi}} \cdot n^{1/2} + 1$$

and then since $T(S) \leq 2 \cdot H(S) + 1$, we have

$$T(S) < \frac{8}{3\pi^{3/2}} \cdot n^{3/2} + \frac{3}{\pi} \cdot n + \frac{13}{3\sqrt{\pi}} \cdot n^{1/2} + 3.$$

A computation then reveals that we have, for example,

$$\begin{aligned} T(S) &< 3.437207165 \cdot n^{3/2} \text{ for all } n \geq 2, \\ T(S) &< 0.8482098513 \cdot n^{3/2} \text{ for all } n \geq 20, \\ T(S) &< 0.5597074838 \cdot n^{3/2} \text{ for all } n \geq 200, \\ T(S) &< 0.5015078204 \cdot n^{3/2} \text{ for all } n \geq 2000, \\ T(S) &< 0.4857746665 \cdot n^{3/2} \text{ for all } n \geq 20000, \end{aligned}$$

which implies

$$
\begin{aligned}
F(S) &> 0.1454669376\sqrt{n} \text{ for all } n \geq 2,\\
F(S) &> 0.5894767660\sqrt{n} \text{ for all } n \geq 20,\\
F(S) &> 0.8933237710\sqrt{n} \text{ for all } n \geq 200,\\
F(S) &> 0.9969934260\sqrt{n} \text{ for all } n \geq 2000,\\
F(S) &> 1.029283811\sqrt{n} \text{ for all } n \geq 20000.
\end{aligned}
$$

Antweiler and Bömer [1] observed that the merit factor of Chu sequences appears empirically to grow like a constant times $\sqrt{n}$, but to the best of the current author's knowledge, the existing literature does not contain proofs that the Chu sequences satisfy bounds of the form $T(S) < Bn^{3/2}$ and $F(S) > C\sqrt{n}$ that hold for all n.

4 Conclusions and Further Questions

We have proved that the Chu sequences satisfy, in an asymptotic sense,

$$
\begin{aligned}
T(S) &\lesssim 0.4788989922 \cdot n^{3/2},\\
F(S) &\gtrsim 1.044061500\sqrt{n}.
\end{aligned}
$$

Numerical evidence suggests the constants in front are not the best constants. If we recall that

$$
T(S) = \sum_{k=1}^{n-1} |c_k|^2 = \sum_{k=1}^{n-1} \sin^2\left(\frac{\pi k^2}{n}\right) \csc^2\left(\frac{\pi k}{n}\right),
$$

then we can calculate $T(n)$ numerically for certain values of n and compare with $n^{3/2}$. Doing this, for example, with n from 1000 to 1024 suggests

$$
\begin{aligned}
T(S) &\approx 0.318 \cdot n^{3/2},\\
F(S) &\approx 1.57\sqrt{n}.
\end{aligned}
$$

The empirical result $F(S) \approx 1.57\sqrt{n}$ was already noted in [1]. Furthermore, it was shown in [8] that certain other sequences, related but not identical to Chu sequences, appear to satisfy $F(S) \approx C\sqrt{n}$ for a larger constant C.

We have shown that the function $T_{\min}(n)$ is bounded above by a constant times $n^{3/2}$, and therefore that the function $F_{\max}(n)$ is bounded below by a constant times $\sqrt{n}$. It would be interesting to know whether this can be improved. If there are polyphase Barker sequences of every length, then $T_{\min}(n)$ grows at most linearly in n.

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