Finding "nonobvious" nilpotent matrices
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There was a midterm recently in the introductory linear algebra course at my university. One subquestion, worth 3 of the midterm's 40 points, was essentially as follows.

*We call a square matrix B **nilpotent** if B^m is equal to the zero matrix for some positive integer m. Prove that if B is a nilpotent matrix, then $\det B = 0$.*

The students had just recently learned the definition and some basic properties of determinants—including the fact that $\det(XY) = \det X \cdot \det Y$ for square matrices X and Y. Thus, an extremely short proof is possible. (Indeed, the proof can be succinctly expressed without words as $B^m = 0 \implies (\det B)^m = \det(B^m) = 0 \implies \det B = 0$.)

However, most of the students failed to find the short proof, and many of them instead went off on strange tangents. In fairness, the students had only very recently learned about determinants, so a fact such as $\det(XY) = \det X \cdot \det Y$, although they had learned it, was not yet an "old friend" to them. Thus, it didn't immediately occur to them to use that fact, and they missed the short proof with its "you see it or you don't" quality.

I suspect that another, more general, phenomenon is also at work here. A great many instructors have noted that undergraduates often have few difficulties with computational matters, but have a lot of trouble with proofs. This includes difficulty in coming up with a proof even when there exists one which is quite brief, with no messy or technical details. It's as though the students, on some level, are under the misapprehension that the only way to prove $\det B = 0$ is to *calculate* $\det B$, by a cofactor expansion or similar means. I've sometimes tried to remedy this by telling linear algebra students, "You don't always have to worry about the *entries* of the matrix. Maybe there's a short proof where you get to completely ignore the individual entries."

In their answers to the midterm question, some students tried to show $\det B = 0$ by showing B must have a row or a column of zeros, which led many of them to make the incorrect claim that any nilpotent matrix must have this property. Perhaps part of the reason is that the most "obvious" examples of nilpotent matrices are things like

$$\begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & z & 0 \end{bmatrix}.$$

The simplest counterexample to the students' claim is perhaps

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},$$

which we ended up writing on many students' papers. A 3 by 3 example is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}.$$

However, those last two matrices have the property that their *square* is the zero matrix. Following the philosophy that it's good to amass a large arsenal of examples and counterexamples, one is naturally led to the following question.

Can we find, for each integer $n \geq 2$, an n by n nilpotent matrix B whose index is exactly n, and which also has the property that none of the matrices $B, B^2, \ldots, B^{n-1}$ have any zero entries?

Here and throughout, when we speak of the "index" of a nilpotent matrix B, we mean the smallest integer m with the property that B^m is the zero matrix. (Incidentally, is it obvious that the index of an n by n nilpotent matrix can be at most n? It looks like it "should" be obvious, and I did manage to find a short proof, but it turned out to be a little trickier than I anticipated. The short proof appears at the end of this article.)

The purpose of this article is to show the answer to the above question is "yes" by exhibiting a family of matrices with integer entries that have the desired property. If we express this problem in the language of linear transformations, it is hardly shocking that such examples exist. For the beginning linear algebra student, however, who may tend to think of multiplication of matrices in a more concrete or computational way, it may come as a bit of a surprise.

I will begin by simply exhibiting the family of matrices I found. If B is the n by n matrix in the sequence

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 & -3 \\ 6 & 1 & 1 & -4 \\ 1 & 6 & 1 & -4 \\ 1 & 1 & 6 & -4 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 & 2 & -4 \\ 7 & 1 & 1 & 1 & -5 \\ 1 & 7 & 1 & 1 & -5 \\ 1 & 1 & 7 & 1 & -5 \\ 1 & 1 & 1 & 7 & -5 \end{bmatrix}, \ldots$$

then B^n is the zero matrix, whereas none of the matrices $B, \ldots, B^{n-1}$ have any zero entries.

Where does such a matrix B come from? If we let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ denote columns 1 through n respectively of the matrix

$$\begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{bmatrix}$$

then B is an integer multiple of the unique matrix that sends $\mathbf{v}_k$ to $\mathbf{v}_{k+1}$ (when $1 \leq k \leq n-1$) and sends $\mathbf{v}_n$ to 0. The rest of the article consists of verifying that in general, this has the desired property.

We let I_n denote the $n \times n$ identity matrix, and we let J_n denote the $n \times n$ matrix whose entries are all 1. Observe that $J_n^2 = nJ_n$. We now define

$$A_n := I_n + J_n = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 2 \end{bmatrix}$$

and, as before, we denote the first, second, $\ldots$, nth column of A_n by $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$, $\mathbf{v}_n$ respectively.

It's not difficult to show A_n is nonsingular. For instance, it's straightforward to directly check that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent. More briefly, we can show

$$A_n^{-1} = \frac{1}{n+1}\big((n+1)I_n - J_n\big) = I_n - \frac{1}{n+1}J_n$$

by observing that

$$\begin{aligned} A_n \cdot \big((n+1)I_n - J_n\big) &= \big(I_n + J_n\big)\big((n+1)I_n - J_n\big) \\ &= (n+1)I_n^2 - I_nJ_n + (n+1)J_nI_n - J_n^2 \\ &= (n+1)I_n - J_n + (n+1)J_n - nJ_n \\ &= (n+1)I_n. \end{aligned}$$

We now define B_n to be the unique $n \times n$ matrix satisfying

$$\begin{aligned} B_n\mathbf{v}_1 &= (n+1)\mathbf{v}_2 \\ B_n\mathbf{v}_2 &= (n+1)\mathbf{v}_3 \\ &\vdots \\ B_n\mathbf{v}_{n-1} &= (n+1)\mathbf{v}_n \\ B_n\mathbf{v}_n &= \mathbf{0} \end{aligned}$$

or equivalently, in matrix form,

$$B_nA_n = (n+1)[\mathbf{v}_2 \ \mathbf{v}_3 \ \cdots \ \mathbf{v}_n \ \mathbf{0}],$$

implying that

$$\begin{aligned} B_n &= [\mathbf{v}_2 \ \mathbf{v}_3 \ \cdots \ \mathbf{v}_n \ \mathbf{0}](n+1)A_n^{-1} \\ &= [\mathbf{v}_2 \ \mathbf{v}_3 \ \cdots \ \mathbf{v}_n \ \mathbf{0}]\big((n+1)I_n - J_n\big). \end{aligned}$$

Now certainly, any column vector $\mathbf{x}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{x} = \sum_{k=1}^{n} c_k\mathbf{v}_k,$$

and we then have

$$B_n^r \mathbf{x} = (n+1)^r \sum_{k=1}^{n-r} c_k v_{k+r} \quad \text{if } 1 \le r \le n-1, \tag{1}$$

$$B_n^n \mathbf{x} = \mathbf{0},$$

implying that B_n is nilpotent of index n.

I claim that if $1 \le r \le n-1$, then none of the entries of B_n^r are zero. To show this, it suffices to show that $B_n^r \mathbf{x}$ has no zero entries when $\mathbf{x}$ is one of the standard basis vectors for $\mathbb{R}^n$ (that is, when $\mathbf{x}$ is one of the columns of I_n).

If $\mathbf{x}$ is one of the columns of I_n, then we have

$$\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n$$

where $[c_1 \; c_2 \; \cdots \; c_n]^T$ is one of the columns of A_n^{-1}, implying that $c_k = n/(n+1)$ for one value of k and $c_k = -1/(n+1)$ for all other values of k. Let's say

$$c_k = \begin{cases} n/(n+1) & \text{if } k = \ell \\ -1/(n+1) & \text{if } 1 \le k \le \ell-1 \text{ or } \ell+1 \le k \le n. \end{cases}$$

We now fix r satisfying $1 \le r \le n-1$, and we wish to show that $B_n^r \mathbf{x}$ has no zero entries.

From (1), we know that

$$B_n^r \mathbf{x} = (n+1)^r \sum_{k=1}^{n-r} c_k \mathbf{v}_{k+r},$$

so it suffices to show $\sum_{k=1}^{n-r} c_k \mathbf{v}_{k+r}$ has no zero entries.

We now observe that

$$\sum_{k=1}^{n-r} c_k \mathbf{v}_{k+r} = \begin{cases} \sum_{k=1}^{n-r} \left(\frac{-1}{n+1}\right) \mathbf{v}_{k+r} =: \mathbf{p} & \text{if } \ell > n-r \\ \mathbf{v}_{\ell+r} + \sum_{k=1}^{n-r} \left(\frac{-1}{n+1}\right) \mathbf{v}_{k+r} =: \mathbf{q} & \text{if } \ell \le n-r. \end{cases}$$

We now note that $\mathbf{p} = \frac{-1}{n+1} \sum_{k=1}^{n-r} \mathbf{v}_{k+r}$, and $\sum_{k=1}^{n-r} \mathbf{v}_{k+r}$ is a vector all of whose entries are either $n-r$ or $n-r+1$, so $\mathbf{p}$ is a vector whose entries are all $-(n-r)/(n+1)$ or $-(n-r+1)/(n+1)$. Thus none of the entries of $\mathbf{p}$ are zero, and since none of the entries of $\mathbf{p}$ are -1 or -2, we also get that none of the entries of $\mathbf{q}$ are zero. QED.

We close this article with the promised short proof that an $n \times n$ nilpotent matrix has index at most n. If B is an $n \times n$ nilpotent matrix of index exactly k, then B^{k-1} is not the zero matrix, and hence there exists a vector $\mathbf{x}$ such that $B^{k-1}\mathbf{x}$ is nonzero. We then claim that the k nonzero vectors

$$\mathbf{x}, B\mathbf{x}, B^2\mathbf{x}, \dots, B^{k-1}\mathbf{x}$$

are linearly independent elements of $\mathbb{R}^n$, which would imply $k \leq n$ as required. If $c_0, \ldots, c_{k-1}$ are scalars satisfying

$$c_0\mathbf{x} + c_1 B\mathbf{x} + \cdots + c_{k-2}B^{k-2}\mathbf{x} + c_{k-1}B^{k-1}\mathbf{x} = \mathbf{0},$$

then we also have the $k - 1$ equations

$$c_0 B\mathbf{x} + c_1 B^2\mathbf{x} + \cdots + c_{k-2}B^{k-1}\mathbf{x} + c_{k-1}B^k\mathbf{x} = \mathbf{0},$$
$$c_0 B^2\mathbf{x} + c_1 B^3\mathbf{x} + \cdots + c_{k-2}B^k\mathbf{x} + c_{k-1}B^{k+1}\mathbf{x} = \mathbf{0},$$

$$\vdots$$

$$c_0 B^{k-2}\mathbf{x} + c_1 B^{k-1}\mathbf{x} + \cdots + c_{k-2}B^{2k-4}\mathbf{x} + c_{k-1}B^{2k-3}\mathbf{x} = \mathbf{0},$$
$$c_0 B^{k-1}\mathbf{x} + c_1 B^k\mathbf{x} + \cdots + c_{k-2}B^{2k-3}\mathbf{x} + c_{k-1}B^{2k-2}\mathbf{x} = \mathbf{0}.$$

But the last of these $k - 1$ equations says $c_0 B^{k-1}\mathbf{x} = \mathbf{0}$, implying $c_0 = 0$. Then, since the previous equation says $c_0 B^{k-2}\mathbf{x} + c_1 B^{k-1}\mathbf{x} = \mathbf{0}$, we can conclude that $c_1 = 0$. Continuing in this way, we can conclude that $c_0, \ldots, c_{k-1}$ must all be zero, establishing linear independence as claimed.