A "minimalist" proof that the primes have density zero

As is standard, let $\pi(x)$ denote the number of primes less than or equal to x. The Prime Number Theorem (PNT) says

$$\pi(x) \sim \frac{x}{\log x}$$

(for us, "log" always denotes natural log). Two consequences of PNT are:

1. the number of primes is infinite
2. $\lim_{x\to\infty} \frac{\pi(x)}{x} = 0$; that is, the primes have "density" zero.

Proofs of PNT tend to be lengthy. But if we just want to prove statements 1 and 2, that can be done more easily. Statement 1 has a classic proof due to Euclid (and there are various other proofs as well). Statement 2 can also be proved relatively quickly by elementary means.

The key "trick", which is not original with me, is to use certain properties of the "middle" binomial coefficient $\binom{2n}{n}$.

We know $\binom{2n}{n} < 4^n$ because the number of n-subsets of a $(2n)$-set is less than the total number of subsets.

Also, we know $\binom{2n}{n} = \frac{(n+1)\cdots 2n}{1\cdots n}$ is an integer. All primes from $n+1$ to $2n$ must appear in its prime factorization (because they appear in the numerator but not the denominator), and each such prime is greater than n. It follows that $\binom{2n}{n} > n^{\pi(2n)-\pi(n)}$. We conclude

$$n^{\pi(2n)-\pi(n)} < 4^n$$

which, taking logs and rearranging, gives

$$\pi(2n) - \pi(n) < \log 4 \cdot \frac{n}{\log n}.$$

This implies

$$\pi(2^k) - \pi(2^{k-1}) < \log 4 \cdot \frac{2^{k-1}}{(k-1)\log 2} = \frac{2^k}{k-1}.$$

If we sum from $k = 2$ to $k = 2m$, the left side telescopes and we get

$$\begin{aligned} \pi(2^{2m}) - \pi(2) &< \frac{2^2}{1} + \cdots + \frac{2^m}{m-1} + \frac{2^{m+1}}{m} + \cdots + \frac{2^{2m}}{2m-1} \\ &< 2^2 + \cdots + 2^m + \frac{2^{m+1} + \cdots + 2^{2m}}{m} \\ &< 2^{m+1} + \frac{2^{2m+1}}{m} \end{aligned}$$

which implies

$$\pi(4^m) = \pi(2^{2m}) < 1 + 2^{m+1} + \frac{2^{2m+1}}{m}.$$

Now given any positive x, there exists a positive integer m with

$$4^{m-1} < x \leq 4^m \qquad \text{and hence } m-1 < \log_4 x \leq m.$$

We then have

$$\pi(x) \leq \pi(4^m) < 1 + 2^{(1+\log_4 x)+1} + \frac{2^{2(1+\log_4 x)+1}}{\log_4 x}$$

which simplifies to

$$\pi(x) < 1 + 4\sqrt{x} + \frac{8x}{\log_4 x}$$

implying

$$\frac{\pi(x)}{x} < \frac{1}{x} + \frac{4}{\sqrt{x}} + \frac{8}{\log_4 x}$$

which approaches 0 as x approaches infinity.